Orienting in mid-air through reconfiguration

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1 Introduction

The goal of this project is to develop control that solves the "falling cat" problem of self righting while falling and lands in a pose that minimizes the effects of the impact. The problem is divided into three parts: reorienting during falling, selection of pose to minimize momentum loss at contact, and behavior of joints during impact to minimize peak contact force. The problem is formulated to produce feedforward controls for a general multi-rigid-body system and applied to a three-link robot.

This work has obvious consequences for autonomous robots by providing control policies for minimizing damage after a fall. In addition, these control problems provide the basis for solving a larger class of non-holonomic momentum conserving systems, such as improved stunts during diving and ice-skating. These control policies form hypotheses for how humans might control posture during a fall and might give insight into how certain falls may lead to greater injury. Furthermore, careful analysis with this framework in mind may drive study into intervention therapies that may reduce injury during fall.

2 Theory

For the case of a system of mass particles that are capable of interacting, but are not acted upon by outside forces leads to an interesting compact solution that relates the amount of "deformation" or local movement change to the global change in orientation. This formulation of the "cat falling" problem is given by these papers:

- Shapere A, Wilczek F. Gauge Kinematics of Deformable Bodies. 1988; HUTP–88/A031

- Montgomery R. Gauge theory of the falling cat. Fields Inst Communications. 1993;1:193–218.

The papers employ Einsteinian notation, so this will be introduced at the beginning and then converted to more traditional engineering vector form. The assumptions for this formulation are the following:

- Angular momentum is conserved.
- Linear momentum is conserved and its motion does not affect rotations.
- No external forces act on the system.
- Internal forces can cause local deformation of the particle system.

2.1 Solution from Gauge Theory

The idea of this section is to develop a *gauge* or measure for the amount of rotation occurs in a system caused by a local deformation when the system is in a state of conserved angular momentum. The basic overview of this derivation is that we imagine that we describe the deformation separate from the orientation and then link them through the constraint of the conserved angular momentum.

Begin with the angular momentum of a system of n particles with masses m_i in an inertial frame where the vectors $\mathbf{u}_i$ and $\dot{\mathbf{u}}_i$ locate their positions in the undeformed, unrotated system:

$$\mathbf{H} = \sum_{i=1}^{n} \mathbf{u}_i \times m_i \dot{\mathbf{u}}_i \tag{1}$$

If the system is not acted upon by external forces, angular momentum is conserved. Therefore, the angular momentum will remain constant even if the location of the particles changes as long as the deformation occurs from internal forces. Therefore, the angular momentum of the undeformed, unrotated system with particles located by $\mathbf{u}_i$ will be identical to the deformed, rotated system with particles located by $\mathbf{v}_i$. This gives the constraint of angular momentum as:

$$\mathbf{H} = \sum_{i=1}^{n} \mathbf{u}_i \times m_i \dot{\mathbf{u}}_i = \sum_{i=1}^{n} \mathbf{v}_i \times m_i \dot{\mathbf{v}}_i \tag{2}$$

Next, ignore the constraint of angular momentum and write the location of deformed particles, $\hat{\mathbf{v}}_i$, using a set of displacement vectors, $\delta\mathbf{s}_i$, that describes an infinitesimal change in local coordinate through internal forces. This gives a relation for the location of the deformed, but not rotated particles:

$$\hat{\mathbf{v}}_i = \mathbf{u}_i + \delta\mathbf{s}_i \tag{3}$$

Now consider the constraint of angular momentum. The deformation from internal forces will result in a system wide rotation $\mathbf{\Omega}$ and the new location of the points, $\mathbf{v}_i$, will be a composite of a global rotation, $\mathbf{\Omega}$, and local deformations, $\hat{\mathbf{v}}_i$:

$$\mathbf{v}_i = \mathbf{\Omega}\hat{\mathbf{v}}_i \tag{4}$$

Now the goal is to relate how a local deformation (the $\hat{\mathbf{v}}_i$ information) relates to the global rotation, $\mathbf{\Omega}$. In fact, the relation is completely defined by the angular momentum constraint (2) and the definition of global rotation (4). The remainder of the derivation will be to manipulate these expressions into a form that solves the global rotation in terms of the local deformations.

Begin by taking the derivative of the relation (4) between the deformed, $\hat{\mathbf{v}}_i$, and deformed-rotated, $\mathbf{v}_i$, particles:

$$\dot{\mathbf{v}}_i = \dot{\mathbf{\Omega}}\hat{\mathbf{v}}_i + \mathbf{\Omega}\dot{\hat{\mathbf{v}}}_i \tag{5}$$

To deal with the derivative of the rotation matrix, it is necessary to introduce a cross product operator and a few identities of the rotation matrix. First, the cross product operator can be written as a skew symmetric matrix (notated as $[\cdot]_\times$), such that:

$$\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a} = [\mathbf{a}]_\times \mathbf{b} = -[\mathbf{b}]_\times \mathbf{a} \tag{6}$$

An identity of the cross product that will become useful is stated:

$$\mathbf{Ra} \times \mathbf{Rb} = [\mathbf{Ra}]_\times \mathbf{Rb} = \det(\mathbf{R})\, \mathbf{R}^{-T} [\mathbf{a}]_\times \mathbf{b} \tag{7}$$

Next, identities of the rotation matrix are introduced. First, recall that rotations are by definition orthogonal and non-scaling, so:

$$\mathbf{\Omega}\mathbf{\Omega}^T = \mathbf{I} \tag{8}$$

$$\det(\mathbf{\Omega}) = 1 \tag{9}$$

$$\mathbf{\Omega}^T = \mathbf{\Omega}^{-1} \tag{10}$$

From taking the derivative of (8) leads to a relation showing that the product $\dot{\mathbf{\Omega}}\mathbf{\Omega}^T$ is skew:

$$\dot{\mathbf{\Omega}}\mathbf{\Omega}^T = -\mathbf{\Omega}\dot{\mathbf{\Omega}}^T = -\left(\dot{\mathbf{\Omega}}\mathbf{\Omega}^T\right)^T \tag{11}$$

Using (11) a vector of angular velocity of the system, ω, can be defined and, through a skew matrix, related to the derivative of the rotation matrix with these identities (see theory of infinitesimal rotations for a more rigorous definition):

$$[\omega]_\times \equiv \dot{\mathbf{\Omega}}\mathbf{\Omega}^T \tag{12}$$

$$\dot{\mathbf{\Omega}} = [\omega]_\times \mathbf{\Omega} \tag{13}$$

$$\mathbf{\Omega}^T [\omega]_\times \mathbf{\Omega} = \left[\mathbf{\Omega}^T \omega\right]_\times \tag{14}$$

$$\mathbf{\Omega}^T \dot{\mathbf{\Omega}} = \left[\mathbf{\Omega}^T \omega\right]_\times \tag{15}$$

With these relations, the momentum constraint (2) is revisited and the deformed-rotated velocity in terms of the deformed velocity and global rotation (5) is substituted into the constraint:

$$\mathbf{H} = \sum_{i=1}^{n} \mathbf{v}_i \times m_i \dot{\mathbf{v}}_i = \sum_{i=1}^{n} \left(\mathbf{\Omega}\hat{\mathbf{v}}_i\right) \times m_i \left(\dot{\mathbf{\Omega}}\hat{\mathbf{v}}_i + \mathbf{\Omega}\dot{\hat{\mathbf{v}}}_i\right) \tag{16}$$

Expanding the cross product in (16) and using the cross product matrix identity (6), gives:

$$\mathbf{H} = \sum_{i=1}^{n} m_i \left([\mathbf{\Omega}\hat{\mathbf{v}}_i]_\times \dot{\mathbf{\Omega}}\hat{\mathbf{v}}_i + [\mathbf{\Omega}\hat{\mathbf{v}}_i]_\times \mathbf{\Omega}\dot{\hat{\mathbf{v}}}_i \right) \tag{17}$$

Then insert an identity matrix of rotation matrices (8) into first term of (17):

$$\mathbf{H} = \sum_{i=1}^{n} m_i \left([\mathbf{\Omega}\hat{\mathbf{v}}_i]_\times \left(\mathbf{\Omega}\mathbf{\Omega}^T\right) \dot{\mathbf{\Omega}}\hat{\mathbf{v}}_i + [\mathbf{\Omega}\hat{\mathbf{v}}_i]_\times \mathbf{\Omega}\dot{\hat{\mathbf{v}}}_i \right) \tag{18}$$

Using the identity for factoring matrices out of a cross product (7) and recognizing that $\det\left(\mathbf{\Omega}\right)\mathbf{\Omega}^{-T} = \mathbf{\Omega}$ further simplifies (18) to:

$$\mathbf{H} = \sum_{i=1}^{n} m_i \mathbf{\Omega} \left([\hat{\mathbf{v}}_i]_\times \mathbf{\Omega}^T \dot{\mathbf{\Omega}}\hat{\mathbf{v}}_i + [\hat{\mathbf{v}}_i]_\times \dot{\hat{\mathbf{v}}}_i \right) \tag{19}$$

Next plug in the rotation matrix identity (15) into (19):

$$\mathbf{H} = \sum_{i=1}^{n} m_i \mathbf{\Omega} \left([\hat{\mathbf{v}}_i]_\times \left[\mathbf{\Omega}^T \omega\right]_\times \hat{\mathbf{v}}_i + [\hat{\mathbf{v}}_i]_\times \dot{\hat{\mathbf{v}}}_i \right) \tag{20}$$

Swapping the arguments of the second cross product in the first term gives:

$$\mathbf{H} = \sum_{i=1}^{n} m_i \mathbf{\Omega} \left(- [\hat{\mathbf{v}}_i]_\times [\hat{\mathbf{v}}_i]_\times \mathbf{\Omega}^T \omega + [\hat{\mathbf{v}}_i]_\times \dot{\hat{\mathbf{v}}}_i \right) \tag{21}$$

Then, rewriting the terms as individuals sums and expanding the second term's cross product gives:

$$\mathbf{H} = \mathbf{\Omega} \sum_{i=1}^{n} m_i \left(- [\hat{\mathbf{v}}_i]_\times [\hat{\mathbf{v}}_i]_\times \right) \mathbf{\Omega}^T \omega + \mathbf{\Omega} \sum_{i=1}^{n} \left(\hat{\mathbf{v}}_i \times m_i \dot{\hat{\mathbf{v}}}_i \right) \tag{22}$$

These sums are now in a recognizable form that give the instantaneous inertia, $\hat{\mathbf{I}}$, and angular momentum, $\hat{\mathbf{H}}$, of the deformed particles:

$$\hat{\mathbf{I}} = - \sum_{i=1}^{n} m_i [\hat{\mathbf{v}}_i]_\times [\hat{\mathbf{v}}_i]_\times \tag{23}$$

$$\hat{\mathbf{H}} = \sum_{i=1}^{n} \hat{\mathbf{v}}_i \times m_i \dot{\hat{\mathbf{v}}}_i \tag{24}$$

Using (23) and (24) in (22) results in a simpler expression:

$$\mathbf{H} = \mathbf{\Omega} \left(\hat{\mathbf{I}} \mathbf{\Omega}^T \omega + \hat{\mathbf{H}} \right) \tag{25}$$

Finally, in the case of free-fall and no initial angular momentum, $\mathbf{H} = 0$, a measure of the change in angle presents itself as, $\mathbf{\Omega}^T \omega$:

$$\mathbf{\Omega}^T \omega = -\hat{\mathbf{I}}^{-1} \hat{\mathbf{H}} \tag{26}$$

Since this is actually a vector expression, it is possible to *lift* the vectors into skew matrices and then write the measure solely in terms of the rotation matrix using (15):

$$\mathbf{\Omega}^T \dot{\mathbf{\Omega}} = - \left[\hat{\mathbf{I}}^{-1} \hat{\mathbf{H}} \right]_\times \tag{27}$$

2.2 Solution from generalized momentum

2.2.1 Angular momentum of a system of rigid bodies

A system of, n, rigid bodies is defined with each body having mass, m_i, center of mass, $\mathbf{c}_i$, rotation with respect to the inertial frame, $\mathbf{R}_i$, angular velocity, ω_i, and moment of inertia about its center of mass, $\mathbf{I}_{\mathbf{c}i}$. The center of mass of the system is defined as $\mathbf{c}$ and each rigid body is located with respect to the system center of mass by a vector $\mathbf{w}_i$:

$$\mathbf{c}_i = \mathbf{w}_i + \mathbf{c} \tag{28}$$

Then the angular momentum of the system with respect to the system's center of mass is:

$$\mathbf{H_c} = \sum_{i=1}^{n} \left(\mathbf{R}_i \mathbf{I_c}_i \mathbf{R}_i^T \omega_i + \mathbf{w}_i \times m_i \dot{\mathbf{w}}_i \right) \tag{29}$$

If the rigid body system is a collection of links in a tree structure located with respect to a root body with, p, generalized coordinates, then the angular momentum, $\mathbf{H}(\mathbf{q}, \dot{\mathbf{q}})$, is a function of the generalized coordinates, q_k and speeds, $\dot{q}_k$. Furthermore the angular momentum is linear with respect to the generalized speeds. Thus, we can define a Jacobian, $\mathbf{J}_H(\mathbf{q})$, that is only a function of the generalized coordinates and relates the partial derivatives of the angular momentum with the generalized speeds:

$$\mathbf{H_c} = \sum_{k=1}^{p} \frac{\partial \mathbf{H_c}}{\partial \dot{q}_k} \dot{q}_k = \mathbf{J}_H \dot{\mathbf{q}} \tag{30}$$

$$\mathbf{J}_{H_{jk}} = \frac{\partial \mathbf{H_c}_j}{\partial \dot{q}_k} \tag{31}$$

Now, this Jacobian may be partitioned into only those components pertaining to generalized speeds that locate the rigid body, $\dot{\mathbf{q}}_o$, and the rest of the rigid body, $\dot{\mathbf{q}}_b$:

$$\mathbf{H_c} = \mathbf{J}_{H_o} \dot{\mathbf{q}}_o + \mathbf{J}_{H_b} \dot{\mathbf{q}}_b \tag{32}$$

Next, if the angular momentum is conserved and exactly zero it is possible to write the generalized speeds of the root body in terms of the speeds of the rest of the body:

$$\dot{\mathbf{q}}_o = -\mathbf{J}_{H_o}^{-1} \mathbf{J}_{H_b} \dot{\mathbf{q}}_b \tag{33}$$

We can write this resulting matrix and note that it should not be dependent on the root body generalized coordinates:

$$\mathbf{\Gamma}(\mathbf{q}_b) = -\mathbf{J}_{H_o}^{-1} \mathbf{J}_{H_b} \tag{34}$$

This gives a measure, that interestingly is independent of the time base. Consider if we multiply through by dt we are left with a pure differential:

$$d\mathbf{q}_o = \mathbf{\Gamma}(\mathbf{q}_b) d\mathbf{q}_b \tag{35}$$

Unfortunately, the right hand side is not easily integrable. A special case for integrating this system is to force all but one generalized speed to be zero, then:

$$\Delta \mathbf{q}_o = \int d\mathbf{q}_o = \int \mathbf{\Gamma}(q_{b1}) dq_{b1} = -\int \mathbf{J}_{H_o}^{-1} \frac{\partial \mathbf{H_c}}{\partial \dot{q}_{b1}} dq_{b1} \tag{36}$$

Another option is if the trajectories of the non-root body generalized coordinates are specified, $q_i = f(t)$ and $\dot{q}_i = df(t)/dt$, it is possible to integrate the expression:

$$\Delta \mathbf{q}_o = \int \mathbf{\Gamma}(t) dt \tag{37}$$

3 Application

The planar three-link robot is used as a specific application of the theory to develop a feedforward control to reorient the robot while falling by moving the leg links. We begin by writing the angular momentum in terms of elements of the angular momentum Jacobian (31). The indices specify the root body = 0, left leg = 1, and right leg = 2:

$$H_c = J_0(q_1, q_2)\dot{q}_0 + J_1(q_1, q_2)\dot{q}_1 + J_2(q_1, q_2)\dot{q}_2 \tag{38}$$

Removing the notation of the coordinate dependency in the Jacobian elements, $J_0 := J_0(q_1, q_2)$, and rewriting the velocity of the root body can be found as:

$$\dot{q}_0 = \frac{H_c}{J_0} - \frac{J_1}{J_0}\dot{q}_1 - \frac{J_2}{J_0}\dot{q}_2 \tag{39}$$

Integrating this directly to get the rotation of the root body gives:

$$\Delta q_0 = \int \left(\frac{H_c}{J_0} - \frac{J_1}{J_0}\dot{q}_1 - \frac{J_2}{J_0}\dot{q}_2 \right) dt \tag{40}$$

Fixing each generalized coordinate sequentially and integrating in steps gives:

$$\Delta q_0 = \int_{t_1}^{t_2} \left(\frac{H_c}{J_0} - \frac{J_1}{J_0}\dot{q}_1 \right) \bigg|_{q_2 = q_2(t_1)} dt + \int_{t_2}^{t_3} \left(\frac{H_c}{J_0} - \frac{J_2}{J_0}\dot{q}_2 \right) \bigg|_{q_1 = q_1(t_2)} dt \tag{41}$$

In the case of zero initial angular momentum this further simplifies to:

$$\Delta q_0 = -\int_{q_1 = q_{11}}^{q_1 = q_{12}} \frac{J_1}{J_0} \bigg|_{q_2 = q_{21}} dq_1 - \int_{q_2 = q_{21}}^{q_2 = q_{23}} \frac{J_2}{J_0} \bigg|_{q_1 = q_{12}} dq_2 \tag{42}$$

Solving these exactly gives the result:

$$\Delta q_0 = f\left(q_{11}, q_{12}, q_{21}, q_{23}\right) \tag{43}$$

A desired Δq_0 can then be solved by searching for the right sequence of $(q_{11}, q_{12}, q_{21}, q_{23})$. By hand I have found a sequence that allows me to start fully stretched out and then move the legs a bit and then return to fully stretched out with a small amount of rotation to the root body. Using letters of the alphabet as cartoons of the configuration, the steps look like: I, U, L, I.